NACA TM 1278

7925

TECH LIBRARY KAFB, NM
0144686

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1278

GENERAL SOLUTION OF PRANDTL'S BOUNDARY-LAYER EQUATION

By W. Mangler

Translation of "Die allgemeine Lösung der Prandtlschen Grenzschichtgleichungen."
Lilienthal-Gesellschaft für Luftfahrtforschung Bericht 141, Oct. 1941

Washington
June 1950

AFMDC
TECHNICAL LIBRARY
APL 2811

319.98/12

TECH LIBRARY KAFB, NM
0144686

NATIONAL ADVISORY COMMITTEE FOR AERONAU...

TECHNICAL MEMORANDUM 1278

GENERAL SOLUTION OF PRANDTL'S BOUNDARY-LAYER EQUATION*

By W. Mangler

Abstract: A method is described by means of which the laminar friction layer at a wall with arbitrary pressure distribution may be calculated from Prandtl's boundary-layer equations.

Outline:

Symbols

x,y	coordinates parallel and vertical to the wall
u,v	velocity components parallel and vertical to the wall
$U(x)$	prescribed velocity distribution
ψ	stream function
U_∞	reference velocity (free-stream velocity)
L	reference length (profile chord or half the profile circumference, respectively)
$Re = \frac{U_\infty L}{\nu}$	Reynolds number

---

*"Die allgemeine Lösung der Prandtlschen Grenzschichtgleichungen." Lilienthal-Gesellschaft für Luftfahrtforschung Bericht 141, Oct. 1941, pp. 3-7.

I. INTRODUCTION

The solutions of Prandtl's boundary-layer equations (reference 1) known so far (compare Howarth's report (reference 2) and K. Schröder's lecture on the Göttingen boundary-layer meeting) represent special solutions inasmuch as they are valid only for special pressure distributions and special initial values. For general application, the problem requires the solution of how a velocity profile prescribed at a point on a solid wall develops along the wall under the influence of a given pressure distribution.

A transformation of the boundary-layer equations, so far probably unknown,[1] will be given below with the aid of which the problem may be traced back to one already known: the solution of the heat-conduction equation. In order to enable, also, calculation of the stagnation point profile, a second transformation is used which by a wholly analogous process also leads to the solution of a linear differential equation. The connection between the new transformation and the transformation into the heat-conduction equation previously indicated by Prandtl (reference 3) and Mises (reference 4) is shown.

By means of the resulting calculation method, the laminar stagnation point profile and the boundary layer on the circular cylinder may be calculated.

II. TRANSFORMATION OF THE BOUNDARY-LAYER EQUATIONS INTO THE HEAT-CONDUCTION EQUATION

In calculating the laminar friction layer on a wing profile for high Reynolds numbers, one may presuppose the radius of curvature of the wall as large in proportion to the dimensions of the boundary layer so that the wall curvature may be neglected. It enters into the calculation only indirectly because the influence of the pressure distribution along the wall, which is impressed on the boundary layer from the outside and which is known from measurements or calculation, is considered.

---

[1]After conclusion of this report the author's attention was called to a report by Piercy and Preston (reference 12) in which a similar transformation of the boundary-layer equations is performed for the case of a constant pressure distribution.

If u and v denote the components of the velocity in the friction layer in x- and y-direction, respectively, that is, parallel or vertical to the wall $y = 0$, ν, the kinematic viscosity of the air, and $U(x)$, the velocity distribution outside of the friction layer to be calculated by means of Bernoulli's equation, u and v must satisfy the force equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

and the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

The latter is to be fulfilled by introduction of the stream function $\psi(x,y)$

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{2a}$$

One may put

$$\psi(x,0) = 0 \tag{3a}$$

As boundary conditions one has at the wall the condition of no slip

$$u(x,0) = 0, \qquad v(x,0) = 0 \tag{3b}$$

and for large distance from the wall the end condition

$$u(x,\infty) = U(x) \tag{3c}$$

In addition, there has to be a suitable initial condition; $u(0,y)$ must be known. If the friction layer begins at a stagnation point $[U(0) = 0]$, the most important case for the wing, one has to put

$$u(0,y) = 0 \tag{4}$$

It is shown that a linear differential equation may be obtained from the quadratic equation (1), by first solving (1) asymptotically for large y, thus linearizing with respect to small values of $U - u$. If L denotes a fixed reference length, U_∞ a constant velocity, and $Re = \frac{U_\infty L}{\nu}$, a Reynolds number, one may write (1) also in the form

$$\left.\begin{aligned}
&Re^{-1}\left(\frac{U}{U_\infty}\right)^{-1} L^2 \frac{\partial^2}{\partial y^2} \frac{U(U - u)}{{U_\infty}^2} - L \frac{\partial}{\partial x} \frac{U(U - u)}{{U_\infty}^2} \\
&\quad + \frac{yU'}{U} L \frac{\partial}{\partial y} \frac{U(U - u)}{{U_\infty}^2} \\
&\quad = L \frac{\partial}{\partial y} \frac{U - u}{U_\infty}\left(\int_0^y \frac{\partial}{\partial x} \frac{U - u}{U_\infty} dy - \frac{yU'}{U} \frac{U - u}{U_\infty}\right) \\
&\quad - \frac{U - u}{U_\infty}\left(L \frac{\partial}{\partial x} \frac{U - u}{U_\infty} - \frac{yU'}{U} L \frac{\partial}{\partial y} \frac{U - u}{U_\infty}\right)
\end{aligned}\right\} \tag{5}$$

For large y the terms of the right side are small compared to those of the left so that one obtains, for large values of y, a linear equation for the function $\frac{U(U - u)}{{U_\infty}^2}$. The latter may be transformed into the heat-conduction equation by passing to new coordinates ξ, η. These are defined by

$$\left.\begin{aligned}
\xi &= \int_0^x \frac{U(x)\ dx}{U_\infty L} \\
\eta &= \frac{yU(x)}{LU_\infty} \sqrt{Re}
\end{aligned}\right\} \tag{6}$$

thus

$$\frac{\partial}{\partial \xi} = \left(\frac{U}{U_\infty}\right)^{-1}\left(L\,\frac{\partial}{\partial x} - \frac{yU'}{U_\infty}\,L\,\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \eta} = \left(\frac{U}{U_\infty}\right)^{-1}\mathrm{Re}^{-1/2}L\,\frac{\partial}{\partial y}$$

If one finally puts

$$\zeta = \left(\frac{y}{x} - \frac{\psi}{Ux}\right)\frac{Ux}{U_\infty L}\left(\frac{U}{U_\infty}\right)^2\sqrt{\mathrm{Re}} \tag{7}$$

and

$$\frac{U^2(x)}{U_\infty^{\,2}} = \bar{a}(\xi) \tag{8}$$

there results

$$\frac{\partial \zeta}{\partial \eta} = \frac{U(U - u)}{U_\infty^{\,2}} \tag{9}$$

and one obtains from (5) the differential equation

$$\frac{\partial^3 \zeta}{\partial \eta^3} - \frac{\partial^2 \zeta}{\partial \xi\,\partial \eta} = \frac{\partial^2 \zeta}{\partial \eta^2}\frac{\partial}{\partial \xi}\left(\frac{\zeta}{\bar{a}}\right) - \frac{\partial \zeta}{\partial \eta}\left[\frac{\partial^2}{\partial \xi\,\partial \eta}\left(\frac{\zeta}{\bar{a}}\right) + \frac{\bar{a}'}{2\bar{a}}\,\frac{\partial}{\partial \eta}\left(\frac{\zeta}{\bar{a}}\right)\right]$$

or written differently

$$\frac{\partial^3\zeta}{\partial\eta^3} - \frac{\partial^2\zeta}{\partial\xi\,\partial\eta} = \frac{1}{\bar{a}(\xi)}\left[\frac{\partial\zeta}{\partial\xi}\frac{\partial^2\zeta}{\partial\eta^2} - \frac{\partial\zeta}{\partial\eta}\frac{\partial^2\zeta}{\partial\xi\,\partial\eta}\right] - \frac{\bar{a}'}{\bar{a}^2}\left[\zeta\frac{\partial^2\zeta}{\partial\eta^2} - \frac{1}{2}\left(\frac{\partial\zeta}{\partial\eta}\right)^2\right] \qquad (10)$$

The boundary conditions now read

$$\zeta(\xi,0) = 0, \qquad \zeta_\eta(\xi,0) = \bar{a}(\xi), \qquad \zeta_\eta(\xi,\infty) = 0 \qquad (11)$$

and the initial condition for the case of the stagnation point is

$$\zeta_\eta(0,\eta) = 0 \qquad (12)$$

The transformation determinant has the value

$$\xi_x\eta_y - \xi_y\eta_x = \frac{U^2(x)}{U_\infty^2}\frac{\sqrt{Re}}{L^2}$$

thus becomes zero at the stagnation point. If there U is proportional to x, ξ becomes proportional x^2 and η proportional to xy so that the straight line $\xi = 0$ corresponds to the straight line $x = 0$ although the correspondence is not a unique, point by point reversible one. Since thereby the boundary conditions and the initial condition are not infringed upon, this singularity does not cause any disturbance. For calculation of the stagnation point profile, however, one must turn to other coordinates (compare section III).

For a general solution of equation (10), one first calculates a first approximation of $\frac{\partial\zeta}{\partial\eta}$ by putting the left side of (10) equal to zero and satisfying the boundary and initial conditions (11), (12). After introducing this first approximation into the right side one determines a second approximation from (10) with known right side. In this manner, the solution can, step by step, be improved by iteration. The essential fact is that the first approximation satisfies both boundary conditions, and completely solves equation (10) above all for large η, where the solutions contain an essential singularity.

The iteration method may also be written as a series development. If ζ_K denotes the difference of the K^{th} and $(K-1)^{th}$ approximation for $\frac{\partial\zeta}{\partial\eta}$, one has

$$\frac{\partial\zeta}{\partial\eta} = \sum_{K=1}^{\infty} \zeta_K, \qquad \zeta = \sum_{K=1}^{\infty} \int_0^{\eta} \frac{\partial\zeta_K}{\partial\eta}\, d\eta \tag{13}$$

with ζ_K satisfying the equation $(K = 1, 2, \ldots ; R_1 \equiv 0)$

$$\begin{aligned}\frac{\partial^2\zeta_K}{\partial\eta^2} - \frac{\partial\zeta_K}{\partial\xi} &= R_K(\xi,\eta) \\ &= \sum_{j=1}^{K-1}\left\{\frac{1}{\bar{a}}\left[\frac{\partial}{\partial\xi}\int_0^{\eta}\zeta_j\, d\eta\, \frac{\partial\zeta_{K-j}}{\partial\eta} - \frac{\partial\zeta_j}{\partial\zeta}\zeta_{K-j}\right]\right. \\ &\quad \left. - \frac{\bar{a}'}{\bar{a}^2}\left[\int_0^{\eta}\zeta_j\, d\eta\, \frac{\partial\zeta_{K-j}}{\partial\eta} - \frac{1}{2}\zeta_j\zeta_{K-j}\right]\right\}\end{aligned} \tag{14}$$

with the boundary conditions

$$\left.\begin{aligned}\zeta_1(\xi,0) &= \bar{a}(\xi) \\ \zeta_K(\xi,0) &= 0(K \geqq 2) \\ \zeta_K(\xi,\infty) &= 0(K \geqq 1)\end{aligned}\right\} \tag{15}$$

and at the stagnation point the initial condition

$$\zeta_K(0,\eta) = 0(K \geqq 1) \tag{16}$$

This is understood at once if equation (14) is summed over K and the resulting double sum of the right side rearranged. Therewith the solution of the boundary-layer equation is traced back completely to a known problem, the treatment of the inhomogeneous heat-conduction equation.

The shearing stress distribution τ in the friction layer is obtained from

$$\frac{\tau}{\rho U_\infty{}^2}\sqrt{Re} = \frac{\partial^2\zeta}{\partial\eta^2} = -\sum_{K=1}^{\infty}\frac{\partial\zeta_K}{\partial\eta}$$

the displacement thickness

$$\delta^* = \int_0^\infty\left(1 - \frac{u}{U}\right)dy$$

from

$$\frac{U(x)\delta^*(x)}{\nu} = \frac{\zeta(\xi,\infty)}{\bar{a}(\xi)}\sqrt{Re} = \sqrt{Re}\sum_{K=1}^{\infty}\int_0^\infty\frac{\zeta_K}{\bar{a}}\,d\eta$$

and the momentum thickness

$$\vartheta = \int_0^\infty\frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

from

$$\frac{U(x)\delta^*(x)}{\nu} - \frac{U(x)\vartheta(x)}{\nu} = \sqrt{Re}\int_0^\infty\left[\frac{\partial}{\partial\eta}\left(\frac{\zeta}{\bar{a}}\right)\right]^2 d\eta$$

III. A SECOND TRANSFORMATION (NEIGHBORHOOD OF THE STAGNATION POINT)

The singularity of the transformation (6) at the stagnation point $(U(x) = xU'(0) + . . .)$ can be avoided if a further transformation is applied to equation (10).

If one puts

$$\left.\begin{aligned} \xi &= \frac{1}{2} c^2 \bar{\xi}^2 \\ \eta &= c\bar{\xi}\bar{\eta} \\ \zeta &= c\bar{\xi}^3 \bar{\zeta} \\ c &= \sqrt{\frac{LU'(0)}{U_\infty}} \end{aligned}\right\} \tag{17}$$

one obtains altogether a transformation which is regular also for $x = 0$. Moreover the new variables

$$\bar{\xi} = \frac{1}{c}\sqrt{\frac{2\int U\,dx}{U_\infty L}}$$

$$\bar{\eta} = \frac{y}{x}\sqrt{\frac{Ux}{\nu}}\sqrt{\frac{Ux}{2\int U\,dx}}$$

$$\bar{\zeta} = \left[\frac{y}{x} - \frac{\psi}{Ux}\right]\sqrt{\frac{Ux}{\nu}}\sqrt{\frac{Ux}{2\int U\,dx}}\,\frac{U^2(x)}{U'(0)2\int U\,dx} \tag{18}$$

thus

$$\frac{\partial \bar{\zeta}}{\partial \bar{\eta}} = \left(1 - \frac{u}{U}\right) \frac{U^2(x)}{U'(0)2\int U\,dx} \tag{19}$$

are more suitable for representation of the laminar profiles than the former ones because they are better adapted to the course of the boundary layer, the thickness of which increases approximately like $\sqrt{x}$. If in addition, one puts

$$\bar{A}(\bar{\xi}) = \frac{\bar{a}(\xi)}{2c^2\xi}, \qquad \bar{A}(0) = 1 \tag{20}$$

the boundary-layer equation reads in these variables

$$\frac{\partial^3 \bar{\zeta}}{\partial \bar{\eta}^3} + \bar{\eta}\frac{\partial^2 \bar{\zeta}}{\partial \bar{\eta}^2} - 2\frac{\partial \bar{\zeta}}{\partial \bar{\eta}} - \bar{\xi}\frac{\partial^2 \bar{\zeta}}{\partial \bar{\xi}\,\partial \bar{\eta}} = \frac{1}{\bar{A}}\left\{\bar{\xi}\left[\frac{\partial \bar{\zeta}}{\partial \bar{\xi}}\frac{\partial^2 \bar{\zeta}}{\partial \bar{\eta}^2} - \frac{\partial \bar{\zeta}}{\partial \bar{\eta}}\frac{\partial^2 \bar{\zeta}}{\partial \bar{\xi}\,\partial \bar{\eta}}\right] + \bar{\zeta}\frac{\partial^2 \bar{\xi}}{\partial \bar{\eta}^2} - \left(\frac{\partial \bar{\zeta}}{\partial \bar{\eta}}\right)^2\right\}$$

$$- \xi\frac{\bar{A}'}{\bar{A}^2}\left\{\bar{\zeta}\frac{\partial^2 \bar{\zeta}}{\partial \bar{\eta}^2} - \frac{1}{2}\left(\frac{\partial \bar{\zeta}}{\partial \bar{\eta}}\right)^2\right\} \tag{21}$$

The boundary conditions are

$$\bar{\zeta}(\bar{\xi},0) = 0$$

$$\bar{\zeta}_{\bar{\eta}}(\bar{\xi},0) = \bar{A}(\bar{\xi})$$

$$\bar{\xi}_{\bar{\eta}}(\bar{\xi},\infty) = 0 \tag{22}$$

As initial condition $\zeta(0,\eta)$ has to be prescribed so that the ordinary differential equation originating from (21) for $\xi = 0$ is satisfied with the boundary conditions (22).

Since, with the aid of (17), Green's function of the linear differential equation for the rectangular region, which is found by putting the left side of (21) equal to zero, also may be guessed, the singular transformation (6) could as well be avoided and the entire calculation performed with the aid of equation (21); however, for the numerical calculation the use of equation (10) is more convenient.

IV. TRANSFORMATION OF THE BOUNDARY-LAYER EQUATIONS INTO THE HEAT-CONDUCTION EQUATION ACCORDING TO PRANDTL AND MISES

Prandtl (reference 3) and Mises (reference 4) have shown that one, likewise, obtains a form of boundary-layer equations related to the heat-conduction equation, if one introduces, instead of the distance from the wall, the stream function ψ as new independent variable

$$\left.\begin{aligned} \xi^* &= \int_0^x \frac{U\,dx}{U_\infty L} = \xi \\ \eta^* &= \frac{\psi}{U_\infty L}\sqrt{Re} = \eta - \frac{\zeta}{a} \end{aligned}\right\} \tag{23}$$

and puts

$$\zeta^* = \frac{U^2 - u^2}{U_\infty^2} = \zeta_0^* \left[1 - \left(1 - \frac{1}{a}\frac{\partial\zeta}{\partial\eta}\right)^2\right] \tag{24}$$

One must then solve the equation

$$\frac{\partial\zeta^*}{\partial\xi^*} = \sqrt{1 - \frac{\zeta^*}{\zeta_0^*}}\,\frac{\partial^2\zeta^*}{\partial\eta^{*2}} \tag{25}$$

with the boundary conditions

$$\left.\begin{aligned} \zeta^*(\xi^*,0) &= \zeta_0^* = \frac{U^2(x)}{U_\infty^2} \\ \\ \zeta^*(\xi^*,\infty) &= 0 \end{aligned}\right\} \tag{26}$$

and a suitably prescribed initial distribution $\zeta^*(0,\eta^*)$. Asymptotically it is, for large η^*, also transformed into the heat-conduction equation, so that a similar iteration method could be used as for equation (10). An indication to that end is to be found in Kármán and Millikan (reference 5); however, the disadvantage of this method is that here a singular point of the solution lies at the wall as well (the second derivative $\frac{\partial^2\zeta^*}{\partial\eta^{*2}}$ there becomes infinitely large), whereas the solution of (10) is regular at the wall and therefore more appropriate for an iteration method.

V. THE NEW CALCULATION METHOD

In order to calculate a laminar friction layer, one has, therefore, to solve the system (14) with the initial and boundary conditions (15), (16). The solutuions can be represented (compare, for instance, Frank-Mises, reference 6, page 872) with the aid of Green's function

$$G(\xi,\eta;x,y) = \frac{1}{\sqrt{\xi - x}}\left(e^{-\frac{(\eta-y)^2}{4(\xi-x)}} - e^{-\frac{(\eta+y)^2}{4(\xi-x)}}\right) \tag{27}$$

for the rectangle

$$0 \leqq x \leqq \xi, \qquad 0 \leqq y \leqq \infty$$

as definite integrals in the following manner

$$\zeta_1(\xi,\eta) = \frac{1}{2\sqrt{\pi}} \int_0^{\xi} \bar{a}(x) G_y(\xi,\eta;x,0)\,dx \tag{28}$$

and for $K \geqq 2$

$$\zeta_K(\xi,\eta) = \frac{1}{2\sqrt{\pi}} \int_0^{\xi} \left(\int_0^{\infty} G(\xi,\eta;x,y)\left(-R_K(x,y)\right)dy \right) dx \tag{29}$$

Unfortunately, the integration in (28) can be carried out generally only when $\bar{a}(\xi)$ is a polynome in ξ. The integration in (29) is even then not generally performable so that numerical methods must be applied for the evaluation of the integrals.

If one puts

$$\left.\begin{aligned} \sigma_1 &= \frac{\eta - y}{2\sqrt{\xi - x}} \\ \sigma_2 &= \frac{\eta + y}{2\sqrt{\xi - x}} \end{aligned}\right\} \tag{30}$$

and denotes

$$\Phi(\sigma) = \frac{2}{\sqrt{\pi}} \int_0^{\sigma} e^{-\alpha^2} d\alpha$$

the Gaussion error function, G may be written in the form

$$G(\xi,\eta;x,y) = \sqrt{\pi}\left(\frac{\partial\Phi(\sigma_1)}{\partial y} - \frac{\partial\Phi(\sigma_2)}{\partial y}\right)$$

Hence, one obtains from (28) and (29) the relations

$$\zeta_1(\xi,\eta) = \int_{x=0}^{\xi} \overline{a}(x)\ d(\Phi(\sigma_1))_{y=0}$$

$$\zeta_K(\xi,\eta) = \int_{x=0}^{\xi} \left(\int_{y=0}^{\infty} (-R_K(x,y))\ d\left(\frac{\Phi(\sigma_1) - \Phi(\sigma_2)}{2}\right)\right) dx \quad (K \geqq 2) \tag{31}$$

Thus one has to plot $\overline{a}$ against

$$\Phi\left(-\frac{\eta}{2\sqrt{\xi - x}}\right)$$

and to integrate graphically, or, respectively, to plot R_K, for fixed x, first against

$$\frac{1}{2}\left(\Phi(\sigma_1) - \Phi(\sigma_2)\right)$$

and to integrate and to plot the integrals obtained once more against x and to integrate them. The derivatives required in the calculation of the R_K values are obtained from $(K \geqq 2)$

$$\frac{\partial \zeta_K}{\partial \eta} = \int_{x=0}^{\xi} \left(\int_{y=0}^{\infty} (-R_K(x,y))\, d\left(\frac{e^{-\sigma_1^2} + e^{-\sigma_2^2}}{2\sqrt{\pi}\sqrt{\xi - x}} \right) \right) dx$$

$$\frac{\partial^2 \zeta_K}{\partial \eta^2} = \int_{x=0}^{\xi} \left(\int_{y=0}^{\infty} (-R_K(x,y))\, d\left(\frac{\sigma_1 e^{-\sigma_1^2} - \sigma_2 e^{-\sigma_2^2}}{2\sqrt{\pi}(\xi - x)} \right) \right) dx$$

$$\frac{\partial \zeta_K}{\partial \xi} = \frac{\partial^2 \zeta_K}{\partial \eta^2} - R_K(\xi,\eta)$$

Corresponding formulas are valid for the derivatives of ζ_1.

Since this calculation has to be carried out for various pairs of values ξ, η, the procedure is rather extensive even though the abscissa functions may be calculated once for all. Since for determination of a velocity profile at a point ξ only the profiles lying shortly ahead need be known more exactly – the profiles lying farther back having little influence on the calculation at the point ξ – information about a few profiles at larger distances before and about several profiles shortly ahead is sufficient for calculating the profile at a point ξ.

No general statements may be made so far concerning the convergence of the iteration method or an estimation of errors. The numerical calculation (compare section VI) proved that the ζ_K values rapidly decrease in the proximity of the stagnation point whereas in the further course of the boundary layer, particularly in the neighborhood of the separation point, the convergence seems to be less favorable.

VI. EXAMPLE

In order to test the iteration method, at first the velocity distribution at the stagnation point was calculated with the aid of the ordinary differential equation resulting from (21) for $\overline{\xi} = 0$. In figure 1, the first approximations are compared with the solution calculated by Hiemenz (reference 7) and Hartree (reference 8). Thus the solution is approximated from one side by the iteration method.

The laminar friction layer on a circular cylinder was calculated as a further example. The potential theoretical velocity distribution (fig. 2)

$$\frac{U}{U_\infty} = 2 \sin \frac{\pi x}{L}$$

not the experimental one according to Hiemenz (reference 7) was taken as a basis because the function $\bar{a}(\xi)$ may then be written as a polynomial in ξ.

$$\bar{a}(\xi) = \frac{U^2}{U_\infty^2} = 4\pi\xi - \pi^2\xi^2$$

The integral ζ_1 may then be calculated in closed form. The first approximation for $\frac{u}{U} = 1 - \frac{\zeta_1}{\bar{a}}$ is plotted in figure 3 for the points $\pi\xi = 0$ (stagnation point), $\pi\xi = 1$, $\pi\xi = 2$ (maximum velocity), $\pi\xi = 2.6$ (location of the laminar separation) and $\pi\xi = 3$ against

$$\bar{\eta} = \frac{y}{x}\sqrt{\frac{Ux}{\nu}}\sqrt{\frac{Ux}{2\int U\,dx}}$$

(compare section III); figure 4 represents the third approximation

$$\frac{u}{U} = 1 - \frac{1}{\bar{a}}\sum_1^3 \zeta_K$$

As a rough estimate shows, the third approximation, in general, still yields slightly too large values for $\frac{u}{U}$; the error is growing with increasing $\pi\xi$. In the neighborhood of the wall, the various approximations yield partly too large, partly too small values; hence, the

point of laminar separation could not be accurately determined for the present. The calculation is to be continued in order to obtain more accurately the profiles for larger ξ values as well.

A few velocity profiles at the velocity maximum $U' = 0$ are plotted for comparison in figure 5. Thus the profile at the flat plate calculated by Blasius (reference 9) almost agrees with the third approximation. Probably the differences decrease still more if further approximations are calculated. One can see that the approximation method according to Pohlhausen (reference 10) also yields relatively satisfactory values. In contrast, the profile calculated according to an approximation method of the author (reference 11) deviates somewhat more strongly, even though the result for the displacement thickness δ^* is practically correct.

A corresponding comparison for several profiles in the proximity of the separation point (fig. 6) shows a relatively good agreement of the third approximation with the approximation method of the author (reference 11), whereas the corresponding Pohlhausen profile and, most of all, the Hartree profile deviate more strongly.

On the basis of these examples and of the cases calculated according to reference 11, where in contrast to the one-parameter method of Pohlhausen the previous history of a profile was at least partly taken into consideration, the statement is permissible that the previous history in the region of the pressure drop behind a stagnation point has little influence on the form of a boundary-layer profile, so that a one-parameter approximation yields usable values. In the region of pressure rise, however, it will – in order to obtain more accurate calculation – probably always be necessary to consider the entire previous pressure distribution, not only the pressure gradient at the respective point.

VII. SUMMARY

The solution of the boundary-layer equations for the laminar flow along a wall with prescribed pressure distributions is traced back to the solution of a system of linear differential equations of the type of the heat conduction equation. Due to the necessarily large expenditure in time, this does not yet represent a method easily applicable in practice; however, one now has the possibility of completely calculating a few characteristic cases in order to estimate and sometimes improve the usefulness of the known approximation methods.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

VIII. REFERENCES

1. Prandtl, L.: Über Flüssigkeitsbewegung bei sehr kleiner Reibung. Verh. des III. Intern. Math. Kongresses, Heidelberg, 1904. (Available as NACA TM 452.)

2. Howarth, L.: Steady Flow in the boundary layer near the Surface of a Cylinder in a Stream. R. & M. No. 1632, British A.R.C., 1935, pp. 1–55.

3. Prandtl, L.: Zur Berechnung der Grenzschichten. Zeitschr. angew. Math. Mech. 18, 1938, pp. 77–82. (Available as NACA TM 959.)

4. V. Mises, R.: Bemerkungen zur Hydrodynamik. Zeitschr. angew. Math. Mech. 7, 428, 1927.

5. V. Kármán, Th., and Millikan, C. B.: On the Theory of Laminar Boundary Layers Involving Separation. NACA Rep. No. 504, 1934, pp. 1–22.

6. Frank, Ph., and V. Mises, R.: Die Differential – und Integralgleichungen der Mechanik und Physik. B. 1, 2. Aufl., Braunschweig 1930.

7. Hiemenz, K.: Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten Kreiszylinder. Dinglers Polytechn. Journal 326, 321, 1911.

8. Hartree, D. R.: On an Equation Occurring in Falkner and Skan's Approximate Treatment of the Equations of the Boundary Layer. Proc. Cambridge Phil. Soc., vol. 33, pt. II, 1937, pp. 223–239.

9. Blasius, H.: Grenzschichten in Flüssigkeiten mit kleiner Reibung. Zeitschr. f. Math. u. Phys. 56, 1, 1908. (Available as NACA TM 1256.)

10. Pohlhausen, K.: Zur näherungsweisen Integration der Differentialgleichung der laminaren Grenzschicht. Zeitschr. f. angew. Math. Mech. 1, pp. 252–268, 1921.

11. Mangler, W.: Ein Verfahren zur Berechnung der laminaren Reibungsschicht. Lilienthal-Gesellschaft Bericht S 10.

12. Piercy, N. A. V., and Preston, J. H.: A Simple Solution of the Flat Plate Problem of Skin Friction and Heat Transfer. Phil. Mag. VII, s. 21, 1936, pp. 995–1005.

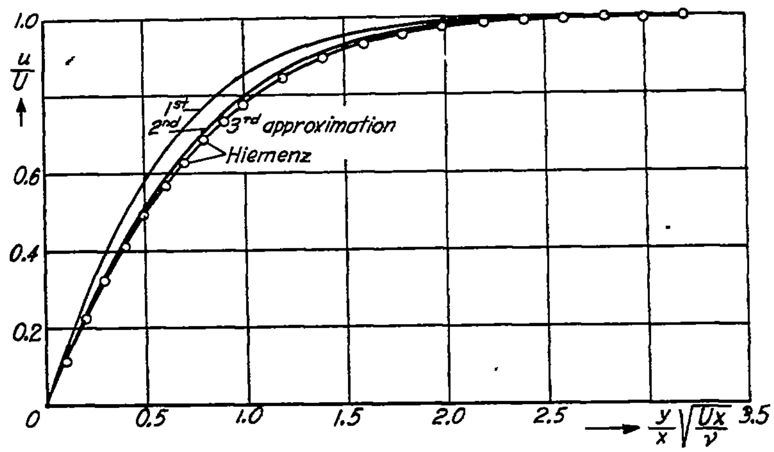

Figure 1.- The velocity profile at the stagnation point.

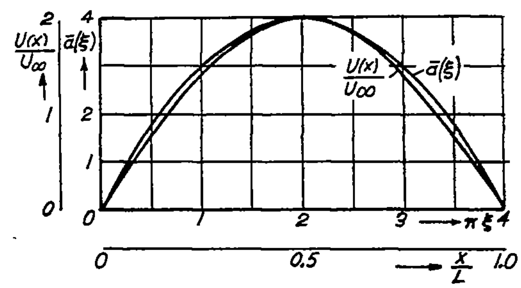

Figure 2.- The velocity distribution $\frac{U}{U_\infty} = 2 \sin \frac{\pi x}{L}$ on the circular cylinder and the function $\bar{a}(\xi)$ pertaining to it.

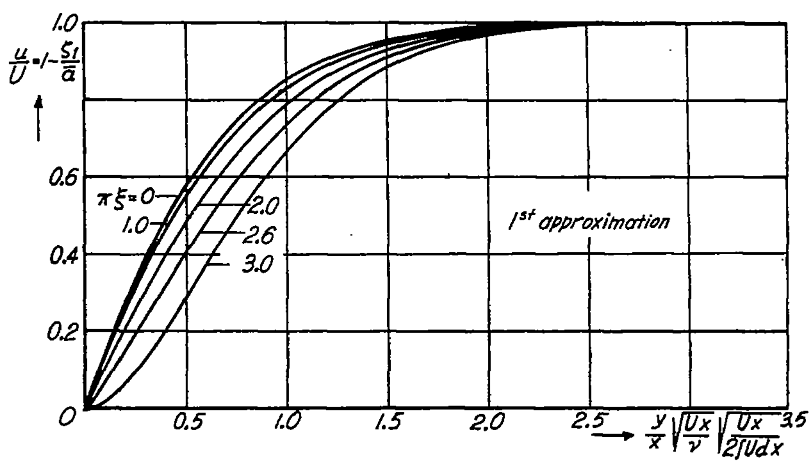

Figure 3.- A few laminar velocity profiles on the circular cylinder (first approximation).

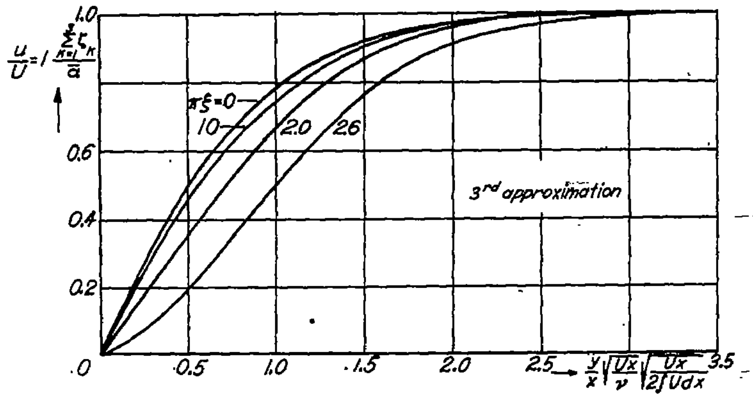

Figure 4.- A few laminar velocity profiles on the circular cylinder (third approximation).

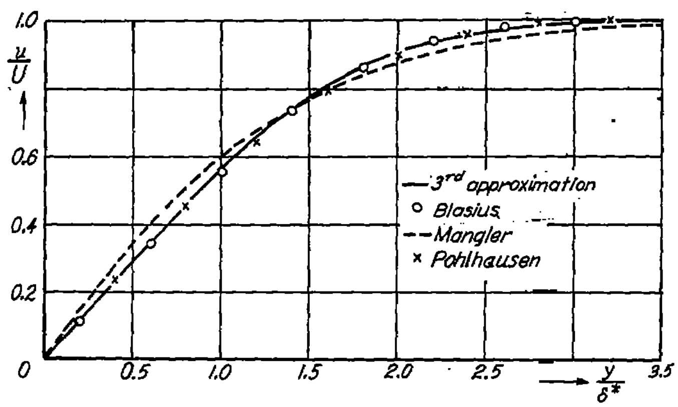

Figure 5.- Comparison of a few laminar velocity profiles at the pressure minimum ($U' = 0$).

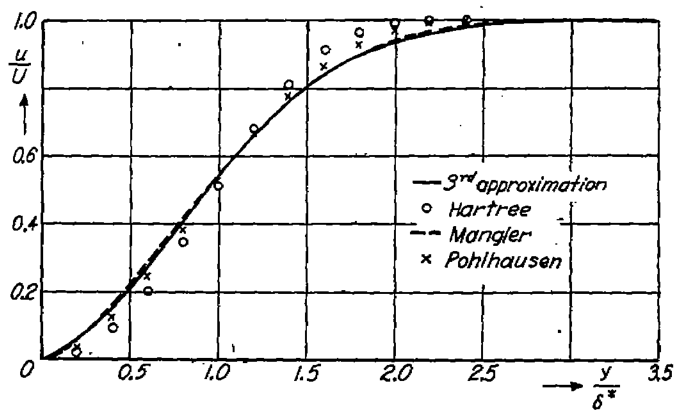

Figure 6.- Comparison of a few laminar velocity profiles in the region of the laminar separation point.

NACA-Langley - 6-26-50 - 1000